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Approximation by random complex polynomials and random rational functions

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Dedicated to the memory of J. Siciak

Abstract. We investigate random compact sets with random functions defined thereon, such as polynomials, rational functions, the pluricomplex Green function and the Siciak extremal function. One surprising consequence of our study is that randomness can be used to 'improve' convergence for sequences of functions.

0. Introduction. We investigate functions and sets with a measurable parameter. For nearly five decades these concepts have been studied in many different mathematical contexts under the names random functions and random sets, even if sometimes the actual randomness played no particular role.

The main objective of this article is to generalize complex approximation theorems to the context of random functions.

Our main results are the following:

- a generalization of Runge's theorem (Theorem 5.7);
- a generalization of the Oka–Weil theorem (Theorem 5.12);
- the image of a random function over a compact set is a random compact set (Theorem 4.7);
- the polynomially and rationally convex hulls of a random compact set are random compact sets (Theorems 4.12 and 4.18);
- the Siciak extremal function and the pluricomplex Green function of a random compact set are random functions (Theorem 4.19);
- a useful convergence theorem that states that from a weak form of convergence we may extract stronger convergence (Theorem 5.3).

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We highlight in more detail three results that illustrate the kind of theorems that we are going to establish. Firstly, a random Runge theorem:

THEOREM 2.4. *Let $K \subset \mathbb{C}$ be an arbitrary non-empty compact set, $(\Omega, \mathcal{A})$ a measurable space and $f : \Omega \times K \to \mathbb{C}$ a random function. Suppose that, for each $\omega \in \Omega$, $f(\omega, \cdot)$ is the restriction to some open neighborhood U_ω of K of some function g_ω holomorphic on U_ω. Then there is a sequence $r_j(\omega, z)$ of random rational functions, pole-free on K, such that*

$$\text{for each } \omega, \quad r_j(\omega, \cdot) \to f(\omega, \cdot) \quad \text{uniformly on } K.$$

The second result, a striking example of the power of the selection method, came as a surprise to us. It shows that the approximation in Theorem 2.4 implies that a stronger approximation is possible.

COROLLARY 5.5. *Let $K \subset \mathbb{C}^n$ be an arbitrary non-empty compact set, $(\Omega, \mathcal{A})$ a measurable space and $f : \Omega \times K \to \mathbb{C}$ a mapping. The following are equivalent:*

(1) *There is a sequence $r_j(\omega, z)$ of random rational functions, pole-free on K, such that*

$$\text{for each } \omega, \quad r_j(\omega, \cdot) \to f(\omega, \cdot) \quad \text{uniformly on } K.$$

(2) *There is a sequence $r_j(\omega, z)$ of random rational functions, pole-free on K, such that*

$$r_j(\omega, z) \to f(\omega, z) \quad \text{uniformly on } \Omega \times K.$$

The third result we wish to highlight generalizes the Oka–Weil theorem.

THEOREM 5.12. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Let K be a random compact set whose range $\{K(\omega) : \omega \in \Omega\}$ consists of at most a countable number of different compact sets. Suppose that K is polynomially convex, i.e., $K(\omega) = \widehat{K}(\omega)$ for all ω. Let f be a generalized random function such that $f(\omega, \cdot)$ is holomorphic in a neighborhood of $K(\omega)$ for each ω, and let ε be a positive measurable function defined on Ω. Then there exists a generalized random polynomial p such that*

$$\|p(\omega, \cdot) - f(\omega, \cdot)\|_{K(\omega)} < \varepsilon(\omega)$$

for all ω outside a measurable set $L \subset \Omega$ such that $\mu(L) = 0$.

The terms used in this theorem will be explained when they first arise.

If $(\Omega, \mathcal{A})$ and $(Z, \mathcal{B})$ are two measurable spaces and X is an arbitrary non-empty set, we shall say that a function $f : \Omega \times X \to Z$ is a *random function* on X if the function $f(\cdot, x)$ is measurable for each $x \in X$. Clearly, if $f : \Omega \times X \to Z$ is a random function and $Y \subset X$, then the restricted mapping $f : \Omega \times Y \to Z$ is a random function. Suppose $f : \Omega \times X \to Z$ is a random function on X and both X and Z are also equipped with topologies. If $f(\omega, \cdot)$ is a continuous function on X for each $\omega \in \Omega$, then we shall say that

f is a *random continuous function* on X. Similarly, we shall speak of random polynomials, random rational functions, random holomorphic functions etc.

We remark that a random function need not be jointly measurable. We thank Eduardo Zeron for bringing to our attention an example given by Sierpiński (see [19, p. 167]). However, a function that is measurable in one variable and continuous in the other (such functions are called *Carathéodory functions*) is jointly measurable (see [22, Theorem 3.1.30]). The same holds for complex-valued functions.

For other studies on the topic of this paper and for historical background, see [2], [4], [5], [7], [8], [15], and references therein.

1. Measurable functions. We use the Borel σ-algebra on $\mathbb{R}^n$ (and $\mathbb{C}^n$ viewed as $\mathbb{R}^{2n}$) to define measurability. To facilitate the reading of this paper, we collect here some well known basic facts that will be used throughout.

PROPOSITION 1.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and Q be a hypercube in $\mathbb{R}^n$. Suppose $f : \Omega \times Q \to \mathbb{C}$ is such that $f(\cdot, x)$ is measurable for all x and $f(\omega, \cdot)$ is Riemann integrable on Q for all $\omega \in \Omega$. Then the function $F(\omega) := \int_Q f(\omega, x)\, \mathrm{d}x$ is measurable.*

Proof. Since $f(\omega, \cdot)$ is Riemann integrable for each ω, we have pointwise convergence of the Riemann sums to F. We easily conclude that F is measurable. ∎

Let K be a compact metric space and let $C(K)$ be the Banach algebra of continuous complex-valued functions on K equipped with the sup norm. We consider $C(K)$ as a measurable space endowed with the σ-algebra $\mathcal{B}$ of Borel subsets of $C(K)$.

PROPOSITION 1.2. *Let $F : \Omega \to C(K)$ be a mapping, and define*

$$f : \Omega \times K \to \mathbb{C}, \qquad (\omega, x) \mapsto F(\omega)(x).$$

Then F is measurable if and only if $f(\cdot, x)$ is measurable for all $x \in K$.

Proof. Suppose F is measurable. For x fixed,

$$f(\omega, x) = F(\omega)(x) = \phi_x(F(\omega)) = (\phi_x \circ F)(\omega),$$

where $\phi_x : C(K) \to \mathbb{C}$ is the evaluation functional at the point x. Thus $f(\cdot, x)$ is the composition of the measurable function F with the continuous function ϕ_x. Therefore $f(\cdot, x)$ is indeed measurable.

We write $\overline{B}_r(g) = \{j \in C(K) : \sup_{x\in K} |j(x) - g(x)| \le r\}$ for the closed ball of radius $r > 0$ around a point g of $C(K)$. Also, we write $f_x^{-1}(E) = \{\omega \in \Omega : f(\omega, x) \in E\}$.

For the converse, since K is compact metric, $C(K)$ is separable. Thus it suffices to show that $F^{-1}\overline{B}_r(g) \in \mathcal{A}$. By hypothesis, K has a countable dense set $\{x_j\}$. Let $B_r = \{z \in \mathbb{C} : |z| \le r\}$, and for $x \in K$ let

$$f_x(\omega) = |g(x) - F(\omega)(x)| = |g(x) - f(\omega, x)|.$$

Then each f_x is measurable and

$$F^{-1}\overline{B}_r(g) = \bigcap_{j=1}^{\infty} f_{x_j}^{-1}(B_r) \in \mathcal{A}. \ \blacksquare$$

COROLLARY 1.3. *Let K be a compact metric space, let $F : \Omega \to C(K, \mathbb{C}^n)$ be a mapping, and define*

$$f : \Omega \times K \to \mathbb{C}^n, \qquad (\omega, x) \mapsto F(\omega)(x).$$

Then F is measurable if and only if $f(\cdot, x)$ is measurable for each $x \in K$.

Proof. For $j = 1, \ldots, n$, let $F_j(\omega)$ be the components of $F(\omega)$, and define

$$f_j : \Omega \times K \to \mathbb{C}, \qquad (\omega, x) \mapsto F_j(\omega)(x).$$

By Proposition 1.2, for each $j = 1, \ldots, n$, the map F_j is measurable if and only if $f_j(\cdot, x)$ is measurable for all $x \in K$. The result follows. $\blacksquare$

2. Runge's theorem for random functions. If U is an open set in $\mathbb{C}$, we say that $f : \Omega \times U \to \mathbb{C}$ is a *random holomorphic function* on U if $f(\omega, \cdot)$ is holomorphic for each $\omega \in \Omega$ and $f(\cdot, z)$ is measurable for each $z \in U$. We define a *random polynomial* p as a random function $p : \Omega \times \mathbb{C} \to \mathbb{C}$ such that $p(\omega, \cdot)$ is a polynomial for each $\omega \in \Omega$.

Consider the following example. Let $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots$ be a partition of Ω into non-empty measurable subsets and, for $n = 0, 1, \ldots$, let a_n be the characteristic function of Ω_n. Then

$$p(\omega, z) = \sum_{n=0}^{\infty} a_n(\omega) z^n$$

is a random polynomial. Indeed, for fixed ω, the function $p(\omega, z)$ is the polynomial z^n, where n is the unique natural number for which $\omega \in \Omega_n$. For fixed z, the function $p(\omega, z)$ is measurable, since it is the pointwise limit of the partial sums and the latter are measurable. Notice that p has infinite degree, although for $\omega \in \Omega_n$, p has degree n, since $p(\omega, z) = z^n$.

For the extended complex plane $\mathbb{C} \cup \{\infty\}$, we use the notation $\overline{\mathbb{C}}$. If $f : \Omega \times X \to \mathbb{C}$ is a random complex function on some set X, then we may also consider f as a random function $f : \Omega \times X \to \overline{\mathbb{C}}$ since, for every Borel set $B \subset \overline{\mathbb{C}}$ and every $x \in X$,

$$\{\omega : f(\omega, x) \in B\} = \{\omega : f(\omega, x) \in B \cap \mathbb{C}\}$$

is measurable, and $B \cap \mathbb{C}$ is a Borel subset of $\mathbb{C}$. In particular, every random polynomial, which by definition is a $\mathbb{C}$-valued random function, can be considered as a $\overline{\mathbb{C}}$-valued random function.

For a compact set $K \subset \mathbb{C}$, let us define a *random rational function pole-free on* K as a random rational function $f : \Omega \times K \to \mathbb{C}$ such that, for each $\omega \in \Omega$, the function $f(\omega, \cdot)$ is a rational function pole-free on K.

Let K be a compact set in $\mathbb{C}^n$ and f be a random function on K that is continuous on K and holomorphic on the interior of K. We say that f is in $R_\Omega(K)$ if there exists a sequence r_j of random rational functions pole-free on K such that, for every $\omega \in \Omega$, $r_j(\omega, \cdot) \to f(\omega, \cdot)$ uniformly on K. Similarly, we say that f is in $R_\Omega^{\mathrm{unif}}(K)$ if for every $\varepsilon > 0$ there exists a random rational function r pole-free on K such that $|r(\omega, z) - f(\omega, z)| < \varepsilon$ for all (ω, z) in $\Omega \times K$.

It would be useful to have sufficient conditions for a random function f to be in $R_\Omega(K)$. Let us call such a result *a Runge theorem for random functions*. Andrus and Brown [1] obtained such a Runge theorem in which the compact set K was also random. We now formulate our first version of a Runge theorem, in which the compact set K is classic (parameter-free) and the approximation is everywhere. Another difference between our presentation and that in [1] is that we have given an explicit definition of random "rational function pole-free on K," whereas in [1] no explicit definition of (the corresponding notion of) random rational function is given.

THEOREM 2.1. *Let U be an open set in $\mathbb{C}$ and $f : \Omega \times U \to \mathbb{C}$ be a random holomorphic function. Let K be a compact subset of U. Then there exists a sequence $R_1, R_2, \ldots$ of random rational functions pole-free on K such that, for each $\omega \in \Omega$,*

$$R_n(\omega, \cdot) \to f(\omega, \cdot) \quad \textit{uniformly on } K.$$

Proof. We can cover K by finitely many disjoint compact sets $Q_1, \ldots, Q_n$, such that each Q_k is contained in U, each Q_k is bounded by finitely many disjoint polygonal curves and $K \subset \bigcup_k \operatorname{int} Q_k$. Let $\Gamma = \bigcup_k \partial Q_k$. By the Cauchy formula, for each $\omega \in \Omega$,

$$f(\omega, z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega, \zeta)}{\zeta - z} \, d\zeta, \qquad \forall z \in K.$$

For $\delta > 0$, partition Γ into $N = N(\delta)$ segments Γ_j of length smaller than δ. For each Γ_j, denote by ζ_j the terminal point of Γ_j. The Riemann sum

$$R(\omega, z) = \sum_{j=1}^{N(\delta)} \frac{1}{2\pi i} \frac{f(\omega, \zeta_j)}{\zeta_j - z} \int_{\Gamma_j} d\zeta = \sum_{j=1}^{N(\delta)} \frac{a_j(\omega)}{\zeta_j - z}$$

is a random rational function pole-free on K. Put

$$\eta(\omega, \delta) := \max\left\{ \frac{1}{2\pi} \left| \frac{f(\omega, \zeta)}{\zeta - z} - \frac{f(\omega, w)}{w - z} \right| : \zeta, w \in \Gamma,\ |\zeta - w| < \delta,\ z \in K \right\}.$$

For all $(\omega, z) \in \Omega \times K$,

$$|f(\omega, z) - R(\omega, z)| < \eta(\omega, \delta) \cdot L(\Gamma),$$

where $L(\Gamma)$ is the length of Γ. It follows from the uniform continuity of $(\zeta, z) \mapsto f(\omega, \zeta)/(\zeta - z)$ on $\Gamma \times K$ that if $\delta = \delta(\omega)$ is sufficiently small, then $\eta(\omega, \delta) < \varepsilon/L(\Gamma)$. Thus,

$$|f(\omega, z) - R(\omega, z)| < \varepsilon, \quad \forall z \in K.$$

Let $\{\delta_n\}_n$ be a sequence of positive numbers decreasing to zero and, for each δ_n, let R_n be a random rational function pole-free on K corresponding as above to δ_n. Then, for each $\omega \in \Omega$,

$$R_n(\omega, \cdot) \to f(\omega, \cdot) \quad \text{uniformly on } K. \ \blacksquare$$

Theorem 2.1 allows us to approximate a random function f on a compact set K, provided there is an open neighborhood U of K such $f(\omega, \cdot)$ is holomorphic on U for all $\omega \in \Omega$. This condition is quite strong and we shall now set the stage for a better version of Runge's theorem.

For an open subset $V \subset \mathbb{C}$, denote by $C^1(V)$ the family of continuously differentiable functions on V, and for $g \in C^1(V)$ and $M \geq 0$ put

$$\|g\|_1 := \sup\big\{\max\{|g(z)|, \|\nabla g(z)\|\} : z \in V\big\},$$
$$C^1(V, M) := \{g \in C^1(V) : \|g\|_1 \leq M\}.$$

Let Γ be a finite union of disjoint smooth curves in V. For a partition $\mathcal{P} = (\zeta_1, \ldots, \zeta_\ell)$ of Γ and a function $g \in C^1(V)$, we denote by $\sum_{g,\mathcal{P}}$ the Riemann sum

$$\sum_{g,\mathcal{P}} := \sum_{\gamma_j} g(\zeta_j) \int_{\gamma_j} \mathrm{d}\zeta,$$

where $\Gamma := \sum_j \gamma_j$ is the decomposition of Γ into arcs induced by the partition $\mathcal{P}$ and ζ_j is the initial point of the arc γ_j.

Lemma 2.2. *Let V, Γ, M be as above. Then, for each $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that, for each $g \in C^1(V, M)$,*

$$\Big|\int_\Gamma g(\zeta) - \sum_{g,\mathcal{P}}\Big| < \varepsilon.$$

Theorem 2.3. *Let U be an open subset of $\mathbb{C}$ and K a compact subset of U. Let $f : \Omega \times U \to \mathbb{C}$ be a random function on U such that, for each ω, there is an open neighborhood U_ω of K in U for which the restriction of $f(\omega, \cdot)$ to U_ω is holomorphic. Then there is a sequence $\{R_k(\omega, z)\}$ of random rational functions pole-free on K such that, for each ω, $R_k(\omega, \cdot) \to f(\omega, \cdot)$ uniformly on K.*

Proof. We may assume that U_ω is relatively compact in U and that $f(\omega,\cdot)$ is holomorphic on a neighborhood of $\overline{U}_\omega$. Thus $f(\omega,\cdot)\in C^1(U_\omega, M_\omega)$ for some finite M_ω.

For $(\omega,\zeta,z)\in\Omega\times U\times\mathbb{C}$, put

$$g_{\omega,z}(\zeta)=\frac{1}{2\pi i}\,\frac{f(\omega,\zeta)}{\zeta-z}.$$

Take Γ_ω in U_ω such that Γ_ω has index 1 with respect to each $z\in K$ and index 0 with respect to each $z\in\mathbb{C}\setminus U_\omega$. Then, by the Cauchy formula,

$$f(\omega,z)=\int_{\Gamma_\omega} g_{\omega,z}(\zeta)\,\mathrm{d}\zeta \qquad \text{for } z\in K.$$

Let U_k, $k=1,2,\ldots$, be a neighborhood basis of K. For each k, choose a $\Gamma_k\subset U_k\setminus K$ such that Γ_k has index 1 with respect to each point of K and index 0 with respect to each $z\in\mathbb{C}\setminus U_k$. Let V_k be a bounded open neighborhood of Γ_k in U_k whose closure is disjoint from K. For each k, there is a finite μ_k such that $1/(\zeta-z)$ as a function of ζ is in $C^1(V_k,\mu_k)$ for each $z\in K$. For each ω, we have $f(\omega,\cdot)\in C^1(U_\omega,M_\omega)$, so $f(\omega,\cdot)\in C^1(U_k,M_\omega)$ for all k such that $U_k\subset U_\omega$. Hence there is a $k(\omega)$ such that, for all $k>k(\omega)$, we have $U_k\subset U_\omega$ and

$$g_{\omega,z}\in C^1(V_k,k\mu_k),\qquad \forall k\ge k(\omega),\ \forall z\in K.$$

In Lemma 2.2, we replace g by $g_{\omega,z}$, V by V_k, Γ by Γ_k, M by $k\mu_k$ and ε by $1/k$. In the Riemann sums, we have the terms

$$g_{\omega,z}(\zeta_j)=\frac{1}{2\pi i}\,\frac{f(\omega,\zeta_j)}{\zeta_j-z},$$

and, by hypothesis, each $f(\cdot,\zeta_j)$ is measurable. Hence, the Riemann sums of $g_{\omega,z}$ are random rational functions pole-free on K, which by the lemma yield the appropriate approximation. ■

Even Theorem 2.3 has a stronger hypothesis than necessary. The following result shows that we can drop the open set U in the statement.

THEOREM 2.4. *Let $K\subset\mathbb{C}$ be an arbitrary non-empty compact set, $(\Omega,\mathcal{A})$ a measurable space and $f:\Omega\times K\to\mathbb{C}$ a random function. Suppose that, for each $\omega\in\Omega$, $f(\omega,\cdot)$ is the restriction to some open neighborhood U_ω of K of some function g_ω holomorphic on U_ω. Then there is a sequence $r_j(\omega,z)$ of random rational functions pole-free on K such that*

$$\textit{for each } \omega,\quad r_j(\omega,\cdot)\to f(\omega,\cdot)\quad \textit{uniformly on } K.$$

We shall establish this result in the next section. In Section 5, we shall prove one of our main results stating that, under appropriate hypotheses, separately uniform convergence implies joint uniform convergence.

3. A measurable extension theorem. In view of Theorem 2.3, to establish Theorem 2.4 it suffices to prove the following measurable extension theorem.

THEOREM 3.1. *Let K be a compact subset of $\mathbb{C}$, let $(\Omega, \mathcal{A})$ be a measurable space, and let $f : \Omega \times K \to \mathbb{C}$ be a function such that:*

(i)′ *$f(\cdot, z)$ is measurable for each $z \in K$,*
(ii)′ *for each $\omega \in \Omega$, there exist an open neighborhood U_ω of K and a function g_ω holomorphic on U_ω such that $g_\omega|_K = f(\omega, \cdot)$.*

Then there exists a function $F : \Omega \times \mathbb{C} \to \mathbb{C}$ such that:

(i) *$F(\cdot, z)$ is measurable for each $z \in \mathbb{C}$,*
(ii) *for each $\omega \in \Omega$, there exist an open neighborhood U_ω of K and a function g_ω holomorphic on U_ω such that $g_\omega|_K = F(\omega, \cdot)$,*
(iii) *$F|_{\Omega \times K} = f$.*

The proof will occupy the rest of this section.

Given a Banach space X, we write X^* for the dual space of X. The following result is well known.

LEMMA 3.2. *Let X be a separable Banach space and let C be a closed convex subset of X. Then there exist sequences $(\phi_n)_{n\geq 1} \subset X^*$ and $(\alpha_n)_{n\geq 1} \subset \mathbb{R}$ such that*

$$C = \bigcap_{n\geq 1} \{x \in X : \operatorname{Re} \phi_n(x) \leq \alpha_n\}. \tag{1}$$

Proof. As X is separable, there is a countable base for its topology consisting of open balls. Let (B_n) be an enumeration of the set of these balls having the property that $B_n \cap C = \emptyset$. By the Hahn–Banach theorem [20, Theorem 3.4(a)], there exist $\phi_n \in X^*$ and $\alpha_n \in \mathbb{R}$ such that

$$\operatorname{Re} \phi_n(x) \leq \alpha_n \quad (x \in C) \quad \text{and} \quad \operatorname{Re} \phi_n(x) > \alpha_n \quad (x \in B_n).$$

Then (1) holds. ∎

Given an open subset U of $\mathbb{C}$, we denote by $A^2(U)$ the Bergman space on U, that is, the subspace of $L^2(U)$ consisting of functions holomorphic on U. It is well known that $A^2(U)$ is a closed subspace of $L^2(U)$, and therefore a Hilbert space. Also, convergence of a sequence in $A^2(U)$ implies uniform convergence on each compact subset of U.

LEMMA 3.3. *Let K be a compact subset of $\mathbb{C}$, let $(\Omega, \mathcal{A})$ be a measurable space, and let $f : \Omega \times K \to \mathbb{C}$ be a function such that:*

- *$f(\cdot, z)$ is measurable for all $z \in K$,*
- *$f(\omega, \cdot) \in C(K)$ for all $\omega \in \Omega$.*

Then $\{\omega : f(\omega, \cdot) \in A^2(U)|_K\} \in \mathcal{A}$ for each open neighborhood U of K.

Proof. Fix U. The restriction map $R_K : A^2(U) \to C(K)$ is linear and continuous, so it is also continuous with respect to the weak topologies on $A^2(U)$ and $C(K)$. The closed unit ball B of $A^2(U)$ is weakly compact, because $A^2(U)$ is a Hilbert space. Therefore $C := R_K(B)$ is weakly compact in $C(K)$. Hence C is closed in $C(K)$. Clearly it is also convex. By Lemma 3.2, there are sequences $(\phi_n) \subset C(K)^*$ and $(\alpha_n) \subset \mathbb{R}$ such that $C = \bigcap_n \{g \in C(K) : \operatorname{Re}\phi_n(g) \le \alpha_n\}$. It follows that

$$\{\omega \in \Omega : f(\omega,\cdot) \in C\} = \bigcap_{n\ge 1} \{\omega \in \Omega : \operatorname{Re}\phi_n(f(\omega,\cdot)) \le \alpha_n\}.$$

By Proposition 1.2 each $\omega \mapsto \phi_n(f(\omega,\cdot))$ is measurable. Therefore

$$\{\omega : f(\omega,\cdot) \in C\} \in \mathcal{A}.$$

Replacing f by f/m, where m is a positive integer, we get

$$\{\omega : f(\omega,\cdot) \in mC\} \in \mathcal{A} \quad (m \ge 1).$$

Finally, we deduce that

$$\{\omega : f(\omega,\cdot) \in A^2(U)|_K\} = \bigcup_{m\ge 1} \{\omega : f(\omega,\cdot) \in mC\} \in \mathcal{A}. \ \blacksquare$$

The next lemma is a sort of measurable identity principle.

LEMMA 3.4. *Let D be a domain in $\mathbb{C}$, let $(\Omega,\mathcal{A})$ be a measurable space, and let $f : \Omega \times D \to \mathbb{C}$ be a function such that $f(\omega,\cdot)$ is holomorphic in D for each $\omega \in \Omega$. Define*

$$M := \{z \in D : f(\cdot,z) \text{ is measurable}\}.$$

If M has a limit point in D, then $M = D$.

Proof. In what follows, we write $f^{(n)}(\omega,z)$ for the nth derivative of $f(\omega,z)$ with respect to z. Define

$$N := \{z \in D : f^{(n)}(\cdot,z) \text{ is measurable for all } n \ge 0\}.$$

We first show that each limit point of M in D belongs to N. Let z^* be such a limit point, say $z^* = \lim_{k\to\infty} z_k$, where $(z_k) \subset M \setminus \{z^*\}$. By Taylor's theorem, for each $n \ge 0$ and each $\omega \in \Omega$, we have

$$f^{(n)}(\omega,z^*)/n! = \lim_{k\to\infty} \frac{f(\omega,z_k) - \sum_{m=0}^{n-1}(f^{(m)}(\omega,z^*)/m!)(z_k-z^*)^m}{(z_k-z^*)^n}.$$

By induction on n, it follows that $\omega \mapsto f^{(n)}(\omega,z^*)$ is measurable for all $n \ge 0$. In other words, $z^* \in N$.

Next, we show N is open in D. Let $z_0 \in D$ and $r := \operatorname{dist}(z_0,\partial D)$. Then, for all $n \ge 0$ and $|z-z_0| < r$ and $\omega \in \Omega$, we have

$$f^{(n)}(\omega,z) = \sum_{m\ge 0} \frac{f^{(n+m)}(\omega,z_0)}{m!}(z-z_0)^m.$$

Hence, if $z_0 \in N$, then $z \in N$ for all z with $|z - z_0| < r$. So N is indeed open in D.

Lastly, we show that N is closed in D. If $z_k \to z_0$ in D, then, for each $n \geq 0$ and $\omega \in \Omega$, we have

$$f^{(n)}(\omega, z_0) = \lim_{k\to\infty} f^{(n)}(\omega, z_k).$$

Hence, if $z_k \in N$ for all k, then also $z_0 \in N$. Thus N is indeed closed in D.

To conclude: if M contains a limit point in D, then N is non-empty, open and closed in D, so, as D is connected, we must have $N = D$. Clearly $N \subset M$, hence also $M = D$. ■

LEMMA 3.5. *Let K be a compact subset of $\mathbb{C}$, let U be an open neighborhood of K, let $(\Omega, \mathcal{A})$ be a measurable space, and let $f : \Omega \times \mathbb{C} \to \mathbb{C}$ be a function such that:*

- *$f(\cdot, z)$ is measurable for all $z \in K$,*
- *for each $\omega \in \Omega$, there exists a function g_ω holomorphic in U such that $g_\omega|_K = f(\omega, \cdot)$.*

Then there exists a function $F : \Omega \times U \to \mathbb{C}$ such that:

- *$F(\cdot, z)$ is measurable for all $z \in U$,*
- *$F(\omega, \cdot)$ is holomorphic in U for all $\omega \in \Omega$,*
- *$F|_{\Omega\times K} = f$.*

Proof. It is enough to construct F on $\Omega \times D$ for each connected component D of U. There are three cases to consider.

If $K \cap D = \emptyset$, then we can simply take $F \equiv 0$.

If $K \cap D$ is non-empty and finite, say $K \cap D = \{z_1, \dots, z_n\}$, then we fix polynomials $p_1, \dots, p_n$ such that $p_j(z_k) = \delta_{jk}$, and define

$$F(\omega, z) := \sum_{j=1}^{n} f(\omega, z_j)p_j(z) \qquad ((\omega, z) \in \Omega \times U).$$

It is easy to see that this F has the required properties.

Finally, if $K \cap D$ is infinite, then, for each $\omega \in \Omega$, we define $F(\omega, \cdot)$ to be the holomorphic extension of $f(\omega, \cdot)$ to D, which exists by hypothesis, and is unique by the identity principle. By Lemma 3.4, $F(\cdot, z)$ is measurable for all $z \in D$. The other two properties required of F are clear. ■

We now have all the ingredients needed to prove Theorem 3.1.

Proof of Theorem 3.1. For each $n \geq 1$, define

$$U_n := \{z \in \mathbb{C} : \operatorname{dist}(z, K) < 1/n\}, \qquad \Omega_n := \{\omega \in \Omega : f(\omega, \cdot) \in A^2(U_n)|_K\}.$$

By Lemma 3.3, we have $\Omega_n \in \mathcal{A}$. Thus, if we set $\mathcal{A}_n := \{A \cap \Omega_n : A \subset \mathcal{A}\}$, then $(\Omega_n, \mathcal{A}_n)$ is a measurable space. By Lemma 3.5, there exists a function $F_n : \Omega_n \times U_n \to \mathbb{C}$ such that

- $F_n(\cdot, z)$ is measurable for each $z \in U_n$,
- $F_n(\omega, \cdot)$ is holomorphic on U_n for each $\omega \in \Omega_n$,
- $F_n = f$ on $\Omega_n \times K$.

Clearly $\Omega_1 \subset \Omega_2 \subset \cdots$, and the hypothesis (ii)$'$ implies that $\bigcup_{n\geq 1} \Omega_n = \Omega$. Define $F : \Omega \times \mathbb{C} \to \mathbb{C}$ by

$$F := \sum_{n\geq 1} F_n 1_{(\Omega_n \setminus \Omega_{n-1}) \times U_n},$$

where, for convenience, we have set $\Omega_0 := \emptyset$. It is routine to check that F satisfies the conclusions of the theorem. ∎

Remark 3.6. The Bergman spaces $A^2(U)$ play only an auxiliary role in this proof. Any other Banach space $H(U)$ of holomorphic functions on U would do just as well, provided that $H(U)$ is reflexive, that $H(U)$ contains all functions that are holomorphic on a neighborhood of $\overline{U}$, and that convergence in $H(U)$ implies uniform convergence on compact sets.

4. Random compact sets. In view of the Oka–Weil theorem, the most important notions in complex approximation in several variables are those of polynomial or rational convexity of compacta. First we state a few basic properties of random compact sets and then show that the image of a compact set by a random continuous function is a random compact set. We also show that the polynomially and rationally convex hulls are transformations that preserve randomness, and that the Siciak extremal function and pluricomplex Green function of a random compact set are random functions.

For a metric space (X, d), we denote by $\mathcal{K}(X)$ the family of compact subsets of X and by $(\mathcal{K}'(X), d_H)$ the space of *non-empty* compact subsets of X equipped with the Hausdorff distance d_H. We recall the following useful property.

Lemma 4.1. *The spaces $\mathcal{K}'(\mathbb{R}^n)$ and $\mathcal{K}'(\mathbb{C}^n)$ are separable.*

Let $g : X \to Y$ be a continuous function. We may extend it to a function $g^{\mathcal{K}'} : \mathcal{K}'(X) \to \mathcal{K}'(Y)$ by setting $g^{\mathcal{K}'}(Q) := g(Q)$ for $Q \in \mathcal{K}'(X)$. The following is well known.

Lemma 4.2. *Let X and Y be metric spaces. If $g : X \to Y$ is continuous, then the extension $g^{\mathcal{K}'} : \mathcal{K}'(X) \to \mathcal{K}'(Y)$ is also continuous.*

Remark 4.3. Since $\mathcal{K}'(X)$ is a metric space, we may also consider it as a measurable space, where the measurable sets are the Borel subsets of $\mathcal{K}'(X)$. As shown in [13, Theorem D.6], we may also characterize $\mathcal{B}(\mathcal{K}'(X))$ as the σ-algebra generated by the sets $\{K \in \mathcal{K}'(X) : K \subset G\}$ where G varies over the open sets of X. Alternatively, the Borel sets can be generated by the sets $\{K \in \mathcal{K}'(X) : K \cap G \neq \emptyset\}$ where again G varies over the open sets of X.

A *random compact set* in X is a measurable function $k : \Omega \to \mathcal{K}'(X)$. If $k_j : \Omega \to \mathcal{K}'(X)$, $j = 1, 2$, are two random compact sets, we denote by $k_1 \cup k_2$ the function $\Omega \to \mathcal{K}'(X)$ defined by $(k_1 \cup k_2)(\omega) := k_1(\omega) \cup k_2(\omega)$ for $\omega \in \Omega$.

Using the characterization of Remark 4.3, one can prove the following lemmas.

Lemma 4.4. *If $k_j : \Omega \to \mathcal{K}'(X)$, $j = 1, 2$, are random compact sets, then $k_1 \cup k_2$ is a random compact set. Also, the countable intersection of random compact sets is a random compact set.*

Lemma 4.5. *Let $\{k_i\}_{i=0}^{\infty}$ be random compact sets. Suppose that for each ω, $k(\omega) := \bigcup_{i=0}^{\infty} k_i(\omega)$ is a compact set. Then k is a random compact set.*

By Corollary 1.3, if X is a compact metric space, we can say that a function $f : \Omega \times X \to \mathbb{C}^n$ is a *random element of* $C(X, \mathbb{C}^n)$ if $f(\omega, \cdot) \in C(X, \mathbb{C}^n)$ for all $\omega \in \Omega$ and $f(\cdot, z)$ is measurable for all $z \in X$.

Lemma 4.6. *Let X be a compact metric space and $f : \Omega \times X \to \mathbb{C}^n$ be a random element of $C(X, \mathbb{C}^n)$. Then, for each $x \in X$, the function*

$$f(\cdot, x) : \Omega \to \mathbb{C}^n, \qquad \omega \mapsto f(\omega, x),$$

is a random complex vector, and the (singleton-valued) function

$$\mathcal{K}^f(\cdot, \{x\}) : \Omega \to \mathcal{K}'(\mathbb{C}^n), \qquad \omega \mapsto \{f(\omega, x)\},$$

is a random compact set.

Proof. The first assertion follows from the definition of a random function.

For brevity, we write $F_x = \mathcal{K}^f(\cdot, \{x\})$. We need to show that F_x is measurable. We use separability arguments. For $W \in \mathcal{K}'(\mathbb{C}^n)$ and $r > 0$, consider the closed ball $\overline{B}_r(W) = \{V \in \mathcal{K}'(\mathbb{C}^n) : d_H(W, V) \leq r\}$ in $\mathcal{K}'(\mathbb{C}^n)$. It is easy to see that every closed ball is closed.

Denote by $\widetilde{x} : C(X, \mathbb{C}^n) \to \mathcal{K}'(\mathbb{C}^n)$ the mapping $g \mapsto \{g(x)\}$ for $g \in C(X, \mathbb{C}^n)$. We then see directly that

$$\begin{aligned} \widetilde{x}^{-1}(\overline{B}_r(W)) &= \{g \in C(X, \mathbb{C}^n) : d_H(g(\{x\}), W) \leq r\} \\ &= \bigcap_{y \in W} \{g \in C(K, \mathbb{C}^n) : d(g(x), y) \leq r\}. \end{aligned}$$

As $\mathbb{C}^n$ is separable and $W \subset \mathbb{C}^n$, W is also separable. Thus there exists a countable dense subset W^* of W, and since d is continuous, we can restrict the intersection to W^*. We also know that each $\{g \in C(X, \mathbb{C}^n) : d(g(x), y) \leq r\}$ is measurable, being closed. Hence, $\widetilde{x}^{-1}(\overline{B}_r(W))$ is measurable, as a countable intersection of measurable sets. We know from Lemma 4.1 that $\mathcal{K}(\mathbb{C}^n)$ is separable. Thus, by a similar argument to that in the proof of Proposition 1.2, every open set in $\mathcal{K}'(\mathbb{C}^n)$ can be expressed as a countable union of closed balls, and we can then generalize to all Borel subsets. Hence, $\widetilde{x}$ is a random function. By hypothesis, f is a random element of $C(X, \mathbb{C}^n)$, so $\omega \mapsto f(\omega, \cdot)$ is measurable. It follows that F_x is measurable, since it is the composition $\widetilde{x}(f(\omega, \cdot))$ of measurable functions:

$$F_x(\omega) = \mathcal{K}^f(\omega, \{x\}) = \{f(\omega, x)\} = \widetilde{x}(f(\omega, \cdot)). \blacksquare$$

Let X be a compact metric space and $g : \Omega \times X \to \mathbb{C}^n$ a random continuous function on X. For $\omega \in \Omega$, we denote $X^g(\omega) = g^{\mathcal{K}'}(\omega, X)$.

THEOREM 4.7. *Suppose X is a compact metric space. If $g : \Omega \times X \to \mathbb{C}^n$ is a random continuous function on X, then $X^g(\omega)$ is a random compact set in $\mathbb{C}^n$.*

Proof. We need only show that X^g is measurable. Let $\{x_1, x_2, \ldots\}$ be a countable (ordered) dense subset of X; let X_j be the set consisting of the first j elements of this set; and let $g_j(\omega, x)$ be the restriction of $g(\omega, x)$ to $\Omega \times X_j$. Clearly, for each j, the function g_j is a random continuous function on X_j. For each $j = 1, 2, \ldots$, we define the set-function

$$F_j : \Omega \to \mathcal{K}'(\mathbb{C}^n), \qquad \omega \mapsto g_j(\omega, X_j).$$

We shall show that F_j is a random compact set.

By Lemma 4.6, the function F_1 is itself measurable (and analogously for all 1-element sets). Moreover, $F_1 : \Omega \to \mathcal{K}'(\mathbb{C}^n)$ is a random continuous compact set.

Denote by $C(X, \mathbb{C}^n)$ the set of continuous functions from X to $\mathbb{C}^n$ and, for $x \in X$, denote by $\widetilde{x} : C(X, \mathbb{C}^n) \to \mathcal{K}'(\mathbb{C}^n)$ the mapping $h \mapsto \{h(x)\}$ for $h \in C(X, \mathbb{C}^n)$. Again, by Lemma 4.6, each $\widetilde{x}_j$ is measurable and then, applying Lemma 4.4 j times, we deduce that F_j is a random compact set. Since $g(\omega, \cdot)$ is continuous for each ω, its extension $g^{\mathcal{K}'}(\omega, \cdot)$ to compact sets is continuous. Since $X_j \to X$, one can pass to the limit:

$$\lim_{j\to\infty} F_j(\omega) = \lim_{j\to\infty} g^{\mathcal{K}'}(\omega, X_j) = g^{\mathcal{K}'}\Big(\omega, \lim_{j\to\infty} X_j\Big) = g^{\mathcal{K}'}(\omega, X) = X^g(\omega).$$

We have shown that the sequence F_j of measurable functions converges pointwise to the function X^g. Since these functions take their values in a metric space, it follows easily that X^g is measurable, as desired. ∎

We define a *compact transformation* to be a function $T : \mathcal{K}'(X) \to \mathcal{K}'(X)$. We say that a compact transformation is *randomness-preserving* if K being a random compact set implies that $T(K)$ is a random compact set. It is important that this property does not depend on a specific structure of Ω, it must work for every measurable space.

Lemma 4.8. *A compact transformation is randomness-preserving if and only if it is a random function.*

Proof. Suppose a compact transformation T is randomness-preserving. This means that for any choice of measurable events set $(\Omega, \mathcal{A})$, the measurability of $\omega \mapsto K(\omega)$ implies the measurability of $\omega \mapsto T(K(\omega))$.

We may thus choose $\Omega = \mathcal{K}'(X)$ and $\mathcal{A} = \mathcal{B}(\mathcal{K}'(X))$. We now look at the identity mapping $I : \Omega \to \mathcal{K}'(X)$ sending K to K. This mapping defines a random compact set, since the identity function is clearly measurable. Thus, as T is randomness-preserving, the mapping $K \mapsto T(I(K)) = T(K)$ is random.

The converse follows directly by composition of measurable functions. ∎

For $K \in \mathcal{K}'(\mathbb{C}^n)$, we denote by $\widehat{K}$ the *polynomially convex hull* of K, defined as

$$\widehat{K} = \Big\{ z \in \mathbb{C}^n : |p(z)| \le \max_{x \in K} |p(x)| \ \forall p \in P(\mathbb{C}^n) \Big\},$$

where $P(\mathbb{C}^n)$ is the set of complex polynomials from $\mathbb{C}^n$ to $\mathbb{C}$. We shall show that the polynomially convex hull of a random compact set is also a random compact set, but first we present some lemmas.

Lemma 4.9. *Let K be a compact subset of $\mathbb{C}^n$ and let $P^{\mathbb{Q}}(\mathbb{C}^n)$ denote the polynomials in $\mathbb{C}^n$ whose coefficients have rational real and imaginary parts. Then*

$$\widehat{K} = \Big\{ z \in \mathbb{C}^n : |p(z)| \le \max_{x \in K} |p(x)| \ \forall p \in P^{\mathbb{Q}}(\mathbb{C}^n) \Big\}. \tag{2}$$

Let W be a measurable space. A *pseudo-random compact set* in X is a mapping

$$f : W \to \mathcal{K}(X),$$

such that pre-images of Borel subsets of $\mathcal{K}'(X)$ are measurable. Here W can be a metric space with the Borel sets, or $W = \Omega$. We see that every random compact set in $\mathbb{C}^n$ is a pseudo-random compact set. If f is a pseudo-random compact set and $f^{-1}(\{\emptyset\})$ is empty, then f is a random compact set (here we see $f^{-1}(\{\emptyset\})$ as the pre-image of the empty compact set, and not as the pre-image of an empty set of compact sets). We see directly that the composition of a measurable function followed by a pseudo-random compact set is a pseudo-random compact set.

A function $f : W \to \mathcal{K}(\mathbb{C}^n)$, where W is a metric space, is *pseudo-continuous* if, for every $w \in W$ such that $f(w) \neq \emptyset$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(w, y) < \delta$ and $f(y) \neq \emptyset$, then $d_H(f(w), f(y)) < \varepsilon$, and moreover $f^{-1}(\emptyset)$ is a measurable set.

Lemma 4.10. *If $\{K_i\}_{i=0}^{\infty}$ is a sequence of pseudo-random compact sets, then $k(\omega) := \bigcap_{i=0}^{\infty} K_i(\omega)$ is a pseudo-random compact set. If $K(\omega) :=$*

$\bigcup_{i=0}^{\infty} K_i(\omega)$ is compact for each ω, then K is a pseudo-random compact set.

Proof. This follows from the fact that we have supposed that the pre-images of Borel subsets are measurable. We may use a similar proof method to the one employed in Lemmas 4.4 and 4.5. ■

LEMMA 4.11. *Let $f : X \to \mathcal{K}(\mathbb{C}^n)$ be a pseudo-continuous compact-valued function. Then it is also a pseudo-random compact set.*

Proof. By our definition of pseudo-continuous function, the restriction $f|_{X\setminus f^{-1}(\emptyset)}$ is continuous. Thus, if O is open in $\mathcal{K}(\mathbb{C}^n)$, then $f^{-1}(O)$ is open relative to $X \setminus f^{-1}(\emptyset)$. From the definition of pseudo-continuity, $f^{-1}(\emptyset)$ is measurable in X, so $X \setminus f^{-1}(\emptyset)$ is also measurable. We see that $f^{-1}(O)$ is a measurable subset of the measurable set $X \setminus f^{-1}(\emptyset)$. Thus, $f^{-1}(O)$ is measurable relative to X. As the open sets generate the Borel subsets, all pre-images of measurable subsets of $\mathcal{K}(\mathbb{C}^n)$ are measurable. ■

THEOREM 4.12. *Let K be a random compact subset of $\mathbb{C}^n$. Then its polynomially convex hull $\widehat{K}$, defined pointwise as $\widehat{K}(\omega) := \widehat{K(\omega)}$, is a random compact set.*

Proof. Let $N_p^K := \{z : |p(z)| \le \max_{x\in K} |p(x)|\}$. Then, by Lemma 4.9,

$$\widehat{K}(\omega) = \bigcap_{p\in P^{\mathbb{Q}}(\mathbb{C}^n)} N_p^{K(\omega)}.$$

The function $|p(\cdot)|$ is continuous, and thus

$$N_p^{K(\omega)} = |p(\cdot)|^{-1}(\overline{B}_{\max_{x\in K(\omega)} |p(x)|}(0))$$

is a closed set, as the pre-image of a closed set. But N_p^K is not necessarily bounded. To solve this, we may write

$$\widehat{K}(\omega) = \bigcup_{i=1}^{\infty} \bigcap_{p\in P^{\mathbb{Q}}(\mathbb{C}^n)} (N_p^{K(\omega)} \cap \overline{B}_i(0)). \tag{3}$$

This is true since for every ω, if i is great enough, the compact set $\widehat{K}(\omega)$ is contained in the ball of radius i. Each $N_p^{K(\omega)} \cap \overline{B}_i(0)$ is compact, since it is the intersection of a closed set with a compact set.

Choose a non-constant polynomial p with rational coefficients and a natural number i. We shall show that the mapping $\omega \mapsto N_p^{K(\omega)} \cap \overline{B}_i(0)$ is a pseudo-random compact set. It will follow from Lemma 4.10 that $\bigcap_{p\in P^{\mathbb{Q}}(\mathbb{C}^n)}(N_p^{K(\omega)} \cap \overline{B}_i(0))$ is a pseudo-random compact set, since the intersection is countable. Then, as we already know that $\widehat{K}(\omega)$ is compact and non-empty, we can apply Lemma 4.10 to prove the theorem.

We consider several auxiliary mappings. By hypothesis, $\omega \mapsto K(\omega)$ is measurable. As $|p(\cdot)|$ is continuous, we know that $K \mapsto |p|(K)$ is continuous from $\mathcal{K}'(\mathbb{C}^n)$ to $\mathcal{K}'(\mathbb{R})$. The mapping $K \mapsto \max\{x : x \in K\}$ between $\mathcal{K}'(\mathbb{R})$ and $\mathbb{R}$ is also continuous. Let $\varepsilon > 0$ and fix $K_0 \in \mathcal{K}'(\mathbb{C}^n)$. Suppose $d_H(K_0, K) < \varepsilon$. Then for each $x \in K_0$ (respectively $x \in K$) there is a $y \in K$ (respectively $y \in K_0$) such that $|x - y| < \varepsilon$. It follows that

$$|\max\{x : x \in K_0\} - \max\{x : x \in K\}| < \varepsilon.$$

Denote $W_p^M := \{z : |p(z)| \le M\}$. We shall now show that the mapping $W : \mathbb{R} \to \mathcal{K}(\mathbb{C}^n)$ from M to $W_p^M \cap \overline{B}_i(0)$ is pseudo-random. Consider, for O open in $\mathbb{C}^n$, the sets

$$H_O := \{K \in \mathcal{K}'(\mathbb{C}^n) : K \cap O \neq \emptyset\},$$

which generate the Borel subsets of $\mathcal{K}'(\mathbb{C}^n)$. Since $W(M) \subset W(N)$ whenever $M \le N$ (by continuity of $|p|$), we have

$$\{M : W(M) \in H_O\} = \{M : W(M) \cap O \neq \emptyset\} \in \{[M_0, \infty), (M_0, \infty), \emptyset\}$$

for some $M_0 \in \mathbb{R}$. Thus W is pseudo-random as a mapping from $\mathbb{R}$ to $\mathcal{K}(\mathbb{C}^n)$.

We can now see that $\omega \mapsto N_p^{K(\omega)} \cap \overline{B}_i(0)$ is in fact the composition

$$\omega \mapsto K(\omega) \mapsto |p(K(\omega))| \mapsto \max\{x \in |p(K(\omega))|\} \mapsto W_p^{\max\{x \in |p(K(\omega))|\}} \cap \overline{B}_i(0)$$

and is therefore pseudo-random. Being always non-empty, it is measurable.

As noted earlier, this proves the theorem. ∎

Corollary 4.13. *The mapping*

$$\mathcal{K}'(\mathbb{C}^n) \to \mathcal{K}'(\mathbb{C}^n), \qquad K \mapsto \widehat{K},$$

is measurable.

Proof. This follows from Lemma 4.8 and Theorem 4.12. ∎

For a random compact set $K : \Omega \to \mathcal{K}'(\mathbb{C}^n)$, we define the *random polynomially convex hull* $M\widehat{K} : \Omega \to \mathcal{K}'(\mathbb{C}^n)$ to be

$$M\widehat{K}(\omega) = \Big\{z : |p(\omega, z)| \le \max_{x \in K(\omega)} |p(\omega, x)|, \forall \text{ random polynomials } p\Big\}.$$

Because of the central role of the polynomially convex hull in the Oka–Weil theorem, the random polynomially convex hull should be investigated in relation to approximation by random polynomials. The following proposition is rather obvious.

Proposition 4.14. *Let K be a random compact set. Then $\widehat{K} = M\widehat{K}$.*

Let $\mathcal{R}\text{-hull}\, K$ denote the rationally convex hull of a compact set K, defined as

$$\mathcal{R}\text{-hull}\, K := \Big\{z \in \mathbb{C}^n : |r(z)| \le \max_{x \in K} |r(x)| \ \forall r \in \mathcal{R}_K(\mathbb{C}^n)\Big\}$$

where $\mathcal{R}_K(\mathbb{C}^n)$ is the set of rational functions from $\mathbb{C}^n$ to $\mathbb{C}$ which are holomorphic over K.

LEMMA 4.15. *Let K be a compact subset of $\mathbb{C}^n$ and let $\mathcal{R}_K^{\mathbb{Q}}(\mathbb{C}^n)$ denote the rational functions in $\mathbb{C}^n$ without singularities on K and whose coefficients have rational real and imaginary parts. Then*

$$\mathcal{R}\text{-hull}\, K = \Big\{ z \in \mathbb{C}^n : |r(z)| \leq \max_{x\in K} |r(x)| \ \forall r \in \mathcal{R}_K^{\mathbb{Q}}(\mathbb{C}^n) \Big\}. \tag{4}$$

The following lemma is a known fact that we simply recall.

LEMMA 4.16. *Let $r : \mathbb{C}^n \to \mathbb{C}$ be a rational function. Then the set $S(r)$ of singularities of r is closed.*

The proof of the next lemma is very simple and left to the reader.

LEMMA 4.17. *Let X be a metric space. Suppose $X = A \cup B$, where A and B are disjoint measurable subsets. Let Y be a metric space and $f : X \to Y$ a function whose restriction f_A to the metric space A is measurable, and whose restriction f_B to B is measurable. Then f is a measurable function.*

THEOREM 4.18. *Let K be a random compact set in $\mathbb{C}^n$. Then its rationally convex hull $\mathcal{R}\text{-hull}\, K$, defined pointwise as*

$$[\mathcal{R}\text{-hull}\, K](\omega) := \mathcal{R}\text{-hull}(K(\omega)),$$

is a random compact set.

Proof. Let r be a rational function and K be a non-empty compact subset of $\mathbb{C}^n$. Set $A_r = \{K \in \mathcal{K}'(\mathbb{C}^n) : K \cap S(r) \neq \emptyset\}$ and $B_r = \mathcal{K}'(\mathbb{C}^n) \setminus A_r$. Then $\mathcal{K}'(\mathbb{C}^n)$ is the disjoint union of A_r and B_r. We define $\mu_r : \mathcal{K}'(\mathbb{C}^n) \to \mathcal{K}'(\mathbb{C}^n)$ by

$$\mu_r(K) := \begin{cases} \widehat{K} & \text{if } K \in A_r, \\ \widehat{K} \cap N_r^K & \text{if } K \in B_r, \end{cases}$$

where $N_r^K = \{z \in \mathbb{C}^n : |r(z)| \leq \max_{x\in K} |r(x)|\}$. We note that, for $K \in B_r$, the set N_r^K is never empty, since it contains K. Since all polynomials are also rational functions, $\mathcal{R}\text{-hull}\, K \subseteq \widehat{K}$. Therefore,

$$\mathcal{R}\text{-hull}\, K = (\mathcal{R}\text{-hull}\, K) \cap \widehat{K} = \bigcap_{r \in \mathcal{R}^{\mathbb{Q}}(\mathbb{C}^n)} \mu_r(K),$$

where $\mathcal{R}^{\mathbb{Q}}(\mathbb{C}^n)$ is the set of all rational functions in $\mathbb{C}^n$ whose coefficients have rational real and imaginary parts. The intersection is countable since $\mathcal{R}^{\mathbb{Q}}(\mathbb{C}^n)$ can be viewed as a subset of $P^{\mathbb{Q}}(\mathbb{C}^n) \times P^{\mathbb{Q}}(\mathbb{C}^n)$.

It is thus sufficient to show that $K \mapsto \mu_r(K)$ is measurable, for then by Lemma 4.4, the mapping $\omega \mapsto \mathcal{R}\text{-hull}\, K(\omega)$ will be measurable.

First of all, let us show that $A_r \subset \mathcal{K}'(\mathbb{C}^n)$ is measurable. We may assume that $A_r \neq \emptyset$ and hence $S(r) \neq \emptyset$. We define a sequence of open subsets of $\mathbb{C}^n$ by

$$O_n = \{z : d(z, S(r)) < 1/n\}.$$

Then $O_{n+1} \subseteq O_n$ and $\bigcap_{n=1}^{\infty} O_n = S(r)$. We have

$$A_r = \bigcap_{n=1}^{\infty} \{K : K \cap O_n \neq \emptyset\}.$$

By our second characterization of measurable sets (see Remark 4.3), this is a countable intersection of measurable sets, and thus measurable. We have shown that A_r and hence also $B_r = \mathcal{K}'(\mathbb{C}_n) \setminus A_r$ are measurable.

The definition of $\mu_r(K)$ depends on whether K is in A_r or B_r. Since $\mathcal{K}'(\mathbb{C}^n) = A_r \cup B_r$ is the union of disjoint measurable sets, in order to check that μ_r is measurable, it suffices, by Lemma 4.17, to show that the restrictions of μ_r to A_r and to B_r are measurable.

By Corollary 4.13, the mapping $K \mapsto \widehat{K}$ is measurable, and thus its restriction to A_r, which is the same as the restriction of μ_r to A_r, is also measurable.

We claim that $K \mapsto \widehat{K} \cap N_r^K$ is measurable over B_r. We first notice that over elements of B_r, the rational function r is in fact continuous. With essentially the same proof as for the polynomially convex hull (which did not use any special properties of polynomials other than continuity), one can show that, over B_r, the mappings $K \mapsto N_r^K \cap \overline{B_i(0)}$ are pseudo-random compact. It then follows from Lemma 4.10, as $K \mapsto \widehat{K}$ is measurable, that each mapping $K \mapsto N_r^K \cap \overline{B_i(0)} \cap \widehat{K}$ is pseudo-random compact. We then see that the mapping

$$K \mapsto N_r^K \cap \widehat{K} = \bigcup_{i=1}^{\infty} N_r^K \cap \overline{B_i(0)} \cap \widehat{K}$$

is pseudo-random compact by Lemma 4.10 and in fact measurable, since $N_r^K \cap \widehat{K}$ is always non-empty.

By Lemma 4.17, the mapping μ_r is measurable. ∎

One can use random compact sets to construct interesting random functions. An important function for complex approximation (see [14]) is the pluricomplex Green function V_K for a non-empty compact set $K \subset \mathbb{C}^n$. It is defined as $V_K(z) := \log \Phi_K(z)$ for $z \in \mathbb{C}^n$, where Φ is the *Siciak extremal function*, defined as

$$\Phi_K(z) := \sup\{|p(z)|^{1/\deg p} : p \in P(\mathbb{C}^n),\ \|p\|_K \leq 1,\ \deg p \geq 1\}.$$

We have the following result (see also [3]).

Theorem 4.19. *The Siciak extremal function and the pluricomplex Green function of a random compact set K are random functions. That is, $(\omega, z) \mapsto \Phi_{K(\omega)}(z)$ and $V_{K(\omega)}(z) = \log \Phi_{K(\omega)}(z)$ are random functions into $\mathbb{R} \cup \{\infty\}$.*

Proof. It is sufficient to show that the Siciak extremal function is a random function. Let p be a polynomial. We define the function

$$g_p(\omega, z) := \begin{cases} |p(z)|^{1/\deg p}, & \|p\|_{K(\omega)} \le 1, \\ 0, & \|p\|_{K(\omega)} > 1. \end{cases}$$

We first prove that this is a random function. Fix $z \in \mathbb{C}^n$. Let M be a measurable subset of $\mathbb{R}$. Then

$$\begin{aligned} \{\omega : g_p(\omega, z) \in M\} = (&\{\omega : \|p\|_{K(\omega)} > 1\} \cap \{\omega : 0 \in M\}) \\ &\cup (\{\omega : \|p\|_{K(\omega)} \le 1\} \cap \{\omega : |p(z)|^{1/\deg p} \in M\}). \end{aligned}$$

The sets $\{\omega : 0 \in M\}$ and $\{\omega : |p(z)|^{1/\deg p} \in M\}$ are always $\emptyset$ or Ω, as the conditions do not depend on ω. The set $\{\omega : \|p\|_{K(\omega)} \le 1\}$ is measurable, because the function $|p(K(\omega))|$ is measurable, since it is the composition of the measurable function $K : \Omega \to \mathcal{K}'(\mathbb{C}^n)$ with the continuous function $\mathcal{K}^{|p|} : \mathcal{K}'(\mathbb{C}^n) \to \mathcal{K}'(\mathbb{R})$. Thus, the set $\{\omega : g_p(\omega, z) \in M\}$ is measurable. This proves that $g_p(\cdot, z)$ is measurable, which means that g_p is a random function.

Let us now return to the Siciak extremal function. We shall prove that it may be written as

$$\Phi_K(z) = \sup\{|p(z)|^{1/\deg p} : p \in P^{\mathbb{Q}}(\mathbb{C}^n),\ \|p\|_K \le 1,\ \deg p \ge 1\},$$

where $P^{\mathbb{Q}}(\mathbb{C}^n)$ is the set of all polynomials with rational coefficients.

To see this, it is sufficient to show that for each $\varepsilon > 0$, each $p \in P(\mathbb{C}^n)$ and each $z \in \mathbb{C}^n$, there is a polynomial $q \in P^{\mathbb{Q}}(\mathbb{C}^n)$ such that

$$\big||p(z)|^{1/\deg p} - |q(z)|^{1/\deg q}\big| < \varepsilon.$$

Suppose that p has degree k. We will choose q of degree k also. Using the fact that if $x > y \ge 0$, then $x^{1/k} - y^{1/k} \le (x-y)^{1/k}$, and the reverse triangle identity $\big||x| - |y|\big| \le |x - y|$, we see that

$$\begin{aligned} \big||p(z)|^{1/\deg p} - |q(z)|^{1/\deg q}\big| &= \big||p(z)|^{1/k} - |q(z)|^{1/k}\big| \\ &\le \big||p(z)| - |q(z)|\big|^{1/k} \le |p(z) - q(z)|^{1/k}. \end{aligned}$$

This reduces the problem to finding q such that $|p(z) - q(z)| < \varepsilon^k$. But we know this is possible, as we can approximate p by choosing q with very close rational coefficients.

But now, by the definition of g_p, we have

$$\Phi_{K(\omega)}(z) = \sup_{p \in P^{\mathbb{Q}}(\mathbb{C}^n) :\, \deg p \ge 1} g_p(\omega, z).$$

We have been able to remove the condition $\|p\|_{K(\omega)} \le 1$ since $g_p(\omega, z)$ always returns 0 if this condition is not fulfilled.

We now prove that this function is measurable, that is, $\Phi_{K(\cdot)}(z)$ is measurable. Consider the subset $(-\infty, a]$ of $\mathbb{R}$. We have

$$\begin{aligned}\{\omega : \Phi_{K(\omega)}(z) \in (-\infty, a]\} &= \{\omega : \Phi_{K(\omega)}(z) \le a\} \\ &= \Big\{\omega : \sup_{p\in P^{\mathbb{Q}}(\mathbb{C}^n):\, \deg p\ge 1} g_p(\omega, z) \le a\Big\} \\ &= \{\omega : g_p(\omega, z) \le a,\ \forall p \in P^{\mathbb{Q}}(\mathbb{C}^n), \deg p \ge 1\} \\ &= \bigcap_{p\in P^{\mathbb{Q}}(\mathbb{C}^n):\, \deg p\ge 1} \{\omega : g_p(\omega, z) \le a\}.\end{aligned}$$

As g_p is a random function, each of the sets $\{\omega : g_p(\omega, z) \le a\}$ is measurable. As $\{p \in P^{\mathbb{Q}}(\mathbb{C}^n) : \deg p \ge 1\}$ is a countable set, we have a countable intersection of measurable sets, which is again a measurable set. Thus $\Phi_{K(\cdot)}(z)$ is a measurable function. ∎

5. Approximation: from parameterwise uniform to jointly uniform. Let K be a random compact set in $\mathbb{C}^n$. In the theory of relations, (or multifunctions, set-valued functions), it is standard to define the graph of K as

$$\operatorname{Gr} K := \{(\omega, z) \in \Omega \times \mathbb{C}^n : z \in K(\omega)\}.$$

If we consider a compact set $K \subset \mathbb{C}^n$ as a random compact set such that $K(\omega) = K$ for all $\omega \in K$, then $\operatorname{Gr} K = \Omega \times K$.

It is also standard to set

$$K^{-1}(z) = \{\omega : z \in K(\omega)\} = \bigcap_{n=1}^{\infty} \{\omega : K(\omega) \cap B_{1/n}(z) \ne \emptyset\}.$$

Note that this does *not* correspond to the pre-image of the compact set $\{z\}$. By the second equality, we see that this set is measurable. We define a *generalized random function* as a function $f : \operatorname{Gr} K \to \mathbb{C}$ (hence $f(\omega, \cdot)$ is defined over $K(\omega)$) such that, for fixed $z \in \mathbb{C}^n$ with $K^{-1}(z) \ne \emptyset$, the function $f(\cdot, z)$ from $K^{-1}(z)$ to $\mathbb{C}$ is measurable. When a compact set K is considered as a random compact set, a generalized random function on $\operatorname{Gr} K = \Omega \times K$ turns out to be just a random function on K.

We say a random compact set K is *uniformly separable* if there exists a countable subset E of $\mathbb{C}^n$ whose intersection with $K(\omega)$ is dense in $K(\omega)$ for every $\omega \in \Omega$. In a way it is a generalization of a random compact set taking countably many values. In fact, we have the following obvious proposition.

PROPOSITION 5.1. *Let K be a random compact set. If*

$$\Big|\bigcup_{\omega\in\Omega} K(\omega)\setminus\overline{\operatorname{int} K(\omega)}\Big| \le \aleph_0, \tag{5}$$

then K is uniformly separable. Moreover, if $K(\omega)$ is the closure of a bounded open subset of $\mathbb{C}^n$ for all $\omega\in\Omega$, then K is uniformly separable.

Note that the case that K takes only countably many values is included in (5). We do not know whether (5) is a necessary condition when K takes uncountably many values.

A generalized random function $f(\omega,z)$ will be said to be a *generalized continuous function* if for each ω the function $f(\omega,\cdot)$ is continuous on $K(\omega)$.

THEOREM 5.2. *Let K be a uniformly separable random compact set. Let f_j be a sequence of generalized continuous functions such that*

$$f_j(\omega,\cdot)\to f(\omega,\cdot)\quad \text{uniformly on } K(\omega),\quad \forall\omega\in\Omega.$$

Then there exists a sequence F_j of generalized continuous functions such that

$$F_j\to f\quad \text{uniformly on } \operatorname{Gr} K$$

and for all $\omega\in\Omega$ and $j\in\mathbb{N}$, there exists $k\in\mathbb{N}$ with $F_j(\omega,\cdot)=f_k(\omega,\cdot)$.

Proof. The function f is well defined as a function from $\operatorname{Gr} K$ to $\mathbb{C}$. For each $z\in\mathbb{C}^n$ such that $K^{-1}(z)\neq\emptyset$, the function $f(\cdot,z)$ is easily seen to be measurable on $K^{-1}(z)$. Thus f is a generalized continuous function.

Denote by $E\subset\mathbb{C}^n$ the countable set that is dense in $K(\omega)$ for all $\omega\in\Omega$. Let g be a generalized continuous function and let $\alpha\in\mathbb{R}$. By continuity of $g(\omega,\cdot)$,

$$\Big\{\omega : \sup_{z\in K(\omega)} |g(\omega,z)|\le\alpha\Big\} = \bigcap_{z\in E}\big(\{\omega\in K^{-1}(z) : |g(\omega,z)|\le\alpha\}\cup\{\omega\notin K^{-1}(z)\}\big).$$

It follows that $\omega\mapsto\sup_{z\in K(\omega)}|g(\omega,z)| = \|g(\omega,\cdot)\|_{K(\omega)}$ is measurable since the right-hand side is a countable intersection of measurable sets.

Let $\varepsilon>0$ and $A_n=\{\omega : \|f_k(\omega,\cdot)-f(\omega,\cdot)\|_{K(\omega)}\le\varepsilon,\ \forall k\ge n\}$. By the previous discussion, these sets are measurable since f_k-f is a generalized continuous function. Then $A_1\subset A_2\subset\cdots$ and $\bigcup_n A_n=\Omega$ by uniform convergence on $K(\omega)$ for all $\omega\in\Omega$. Let $A_0=\emptyset$ and $A_n'=A_n\setminus A_{n-1}$. We consider the measurable function $\varphi:\Omega\to\mathbb{N}$ given by

$$\varphi=\sum_n n\chi_{A_n'}.$$

Define $F(\omega, z) = f_{\varphi(\omega)}(\omega, z)$. Let U be a measurable subset of $\mathbb{C}$ and fix $z \in \mathbb{C}^n$. Then

$$\{\omega \in K^{-1}(z) : F(\omega, z) \in U\} \\ = \bigcup_{k\in\mathbb{N}} \{\omega \in K^{-1}(z) : f_k(\omega, z) \in U\} \cap \{\omega : \varphi(\omega) = k\}$$

and therefore F is a generalized random continuous function by the measurability of f_k and φ. Also, by construction,

$$|f(\omega, z) - F(\omega, z)| = |f(\omega, z) - f_{\varphi(\omega)}(\omega, z)| \le \varepsilon,$$

and so, by taking a sequence decreasing to 0, we can construct a sequence $\{F_j\}$ with the desired properties. ■

In particular, we can apply the previous theorem to families of functions. For each non-empty compact set $K \subset \mathbb{C}^n$, let $A(K)$ be a family of continuous functions on K. Thus,

$$A : \mathcal{K}'(\mathbb{C}^n) \to \mathcal{P}(C(K)).$$

Suppose further that if $Q \subset K$ is a compact set and $f \in A(K)$, then $f|_Q \in A(Q)$. For example, $A(K)$ can be the family of holomorphic, polynomial, or harmonic functions on K, or the family of rational functions having no poles on K. For a random compact set $K(\omega)$, we define a *generalized $A(K)$-random function* as a generalized random function $f : \operatorname{Gr} K \to \mathbb{C}$ such that $f(\omega, \cdot) \in A(K(\omega))$ for all $\omega \in \Omega$.

Suppose $K : \Omega \to \mathcal{K}'(\mathbb{C}^n)$ is a measurable compact subset of $\mathbb{C}^n$, and $K \mapsto A(K)$, for $K \in \mathcal{K}'(\mathbb{C}^n)$, is as above. Denote by $A_{[\Omega]}(K)$ the set of generalized $C(K)$-random functions $f : \operatorname{Gr} K \to \mathbb{C}$ for which there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of generalized $A(K)$-random functions such that, for each $\omega \in \Omega$, $f_j(\omega, \cdot) \to f(\omega, \cdot)$ uniformly on $K(\omega)$. Denote by $A_{[\Omega]}^{\mathrm{unif}}(K)$ the set of generalized $C(K)$-random functions $f : \operatorname{Gr} K \to \mathbb{C}$ for which there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of generalized $A(K)$-random functions such that $f_j \to f$ uniformly on $\operatorname{Gr} K$.

The following theorem is a direct consequence of the previous one.

Theorem 5.3. *Suppose $K : \Omega \to \mathcal{K}'(\mathbb{C}^n)$ is a uniformly separable random compact subset of $\mathbb{C}^n$ and $K \mapsto A(K)$, for $K \in \mathcal{K}'(\mathbb{C}^n)$, is as above. Then $A_{[\Omega]}^{\mathrm{unif}}(K) = A_{[\Omega]}(K)$.*

Denote by $A_\Omega(K)$ the set of $C(K)$-random functions $f : \Omega \times K \to \mathbb{C}$ for which there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of $A(K)$-random functions such that, for each $\omega \in \Omega$, $f_j(\omega, \cdot) \to f(\omega, \cdot)$ uniformly on K. Denote by $A_\Omega^{\mathrm{unif}}(K)$ the set of $C(K)$-random functions $f : \Omega \times K \to \mathbb{C}$ for which there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of $A(K)$-random functions such that $f_j \to f$ uniformly on $\Omega \times K$.

COROLLARY 5.4. *If $K \mapsto A(K)$, for $K \in \mathcal{K}'(\mathbb{C}^n)$, is as above, then $A_\Omega^{\mathrm{unif}}(K) = A_\Omega(K)$.*

The following result came as a surprise, since K is arbitrary and we do not assume that f is defined in a neighborhood of K.

COROLLARY 5.5. *Let $K \subset \mathbb{C}^n$ be a non-empty compact set, $(\Omega, \mathcal{A})$ a measurable space and let $f : \Omega \times K \to \mathbb{C}$ be a mapping. The following are equivalent:*

(1) *There is a sequence $r_j(\omega, z)$ of random rational functions, pole-free on K, such that*
$$\text{for each } \omega, \quad r_j(\omega, \cdot) \to f(\omega, \cdot) \quad \text{uniformly on } K.$$
(2) *There is a sequence $r_j(\omega, z)$ of random rational functions, pole-free on K, such that*
$$r_j(\omega, z) \to f(\omega, z) \quad \text{uniformly on } \Omega \times K.$$

Proof. The constant mapping $K : \Omega \to \mathcal{K}'(\mathbb{C}^n)$ given by $\omega \mapsto K$ is a uniformly separable random compact subset of $\mathbb{C}^n$. ∎

The following particular case is worth a separate statement and is one of our main results.

THEOREM 5.6. $R_\Omega^{\mathrm{unif}}(K) = R_\Omega(K)$.

Theorem 2.4 combined with Theorem 5.6 yields the following rather strong Runge theorem.

THEOREM 5.7. *Let f satisfy the hypotheses of Theorem 2.4. Then $f \in R_\Omega^{\mathrm{unif}}(K)$.*

While the function classes $K \mapsto A(K)$ on compact sets considered in Corollary 5.4 are quite general, covering, for example, many levels of smooth functions, it is desirable to have a similar result for function classes on open sets, in order to include Hardy spaces and other important function classes defined on open sets. From Corollary 5.4, we shall deduce an analogous corollary for approximation on compact subsets of an open set. Suppose $U \mapsto A(U)$ for U open in $\mathbb{C}^n$ is a hereditary class of continuous functions, in the sense that if V is an open subset of U and $f \in A(U)$, then $f|_V \in A(V)$. For a compact set $K \subset \mathbb{C}$, let $A(K)$ consist of those continuous functions f on K for which there is an open neighborhood U of K and $F \in A(U)$ such that $F|_K = f$. Then $K \mapsto A(K)$ satisfies the hypotheses of Corollary 5.4, so from that corollary we deduce the following.

COROLLARY 5.8. *Suppose $U \mapsto A(U)$ is as above and $f \in A(U)$. If there exists an exhaustion $K_1 \subset K_2 \subset \cdots$ of U and, for every j and every ω, a sequence $f_{k,j}$ of $A(K_k)$-random functions such that $f_{k,j}(\omega, \cdot) \to f(\omega, \cdot)$*

uniformly on K_k, then there exists a sequence f_k of $A(K_k)$-random functions such that

$$|f_k(\omega,z) - f(\omega,z)| < 1/k, \qquad \forall \omega \in \Omega,\ \forall z \in K_k,\ \forall k.$$

Proof. Apply Corollary 5.4 to each $A(K_k)$ and take a diagonal sequence. ∎

Corollary 5.9. *Let U be an open set in $\mathbb{C}$ and $f : \Omega \times U \to \mathbb{C}$ be a random holomorphic function. Let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of U by compact subsets and $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers. Then there exists a sequence $R_1, R_2, \ldots$ such that each R_n is a random rational function pole-free on K_n, and*

$$|R_n(\omega,z) - f(\omega,z)| < \varepsilon_n \quad \text{for all } (\omega,z) \in \Omega \times K_n.$$

Proof. From Theorems 2.1 and 5.6, for each n, there is a sequence $R_{n,j}$ of random rational functions pole-free on K_n such that $R_{n,j} \to f$ uniformly on $\Omega \times K_n$. Now take an appropriate diagonal sequence. ∎

Following the method of [1] very closely, we shall now prove an "almost everywhere" version of a measurable Oka–Weil theorem. We first need the following lemmas.

Lemma 5.10. *The set $P(k,m)$ of polynomials in $\mathbb{C}^n$ of degree at most k and coefficients bounded by m (in norm) is closed in $C(K,\mathbb{C})$ for every non-empty compact set $K \subseteq \mathbb{C}^n$.*

Proof. Fix a non-empty compact set K. Choose a sequence p_i of polynomials in $P(k,m)$ converging uniformly to a function $f \in C(K,\mathbb{C})$. We wish to show f is in the restriction of $P(k,m)$ over K.

We denote by d_k the total number of possible terms in such polynomials. Then

$$d_k = \sum_{\ell=0}^{k} \binom{n+\ell-1}{n-1}.$$

We denote by $a_{i,\alpha}$, where $\alpha \in \mathbb{N}^n$, the coefficient of z^α. We may then form the d_k-tuplets of coefficients (by choosing an ordering) $a_i \in \mathbb{C}^{d_k}$ for p_i. To simplify our work, we may endow $\mathbb{C}^{d_k}$ with the max norm:

$$\|a_i\|_\infty = \max_\alpha |a_{i,\alpha}|.$$

The $a_{i,\alpha}$ are bounded in norm by m. Thus the a_i are all in the ball $\overline{B}_m(0) \subset \mathbb{C}^{d_k}$, which is a compact set, and thus the sequence $\{a_i\}_{i=0}^\infty$ has a subsequence $\{a'_i\}_{i=0}^\infty$ that converges to a point $b \in \mathbb{C}^{d_k}$. We denote the associated subsequence of polynomials by $\{p'_i\}_{i=0}^\infty$.

Since p_i converges to f, so does p'_i. But p'_i converges on all compact sets to the polynomial $p(z) = \sum_{|\alpha|\le k} b_\alpha z^\alpha$. Thus, $f = p|_K$. ∎

LEMMA 5.11. *The set $P(\mathbb{C}^n)$ of polynomials over $\mathbb{C}^n$ is a Suslin subset of $C(K,\mathbb{C})$ for every non-empty compact set $K \subseteq \mathbb{C}^n$.*

Proof. Firstly, we have

$$P(\mathbb{C}^n) = \bigcup_{m=0}^{\infty} \bigcup_{k=0}^{\infty} P(k,m).$$

If we fix a compact set $K \subset \mathbb{C}^n$, then this is a countable union of sets each of which is closed relative to $C(K,\mathbb{C})$, by Lemma 5.10. Thus $P(\mathbb{C}^n)|_K$ is a Borel subset of $C(K,\mathbb{C})$.

We have already shown in the proof of Proposition 1.2 that $C(K,\mathbb{C})$ is separable. It is also complete, as a Cauchy sequence under the sup norm converges uniformly, and thus converges toward a continuous function. Thus $C(K,\mathbb{C})$ is a Polish space. It is known that every Borel subset of a Polish space is Suslin. ∎

THEOREM 5.12. *Let $(\Omega,\mathcal{A},\mu)$ be a σ-finite measure space. Let K be a random compact set whose range $\{K(\omega) : \omega \in \Omega\}$ consists of at most a countable number of different compact sets. Suppose that K is polynomially convex, i.e., $K(\omega) = \widehat{K}(\omega)$ for all ω. Let f be a generalized random function such that $f(\omega,\cdot)$ is holomorphic in a neighborhood of $K(\omega)$ for each ω, and let ε be a positive measurable function defined on Ω. Then there exists a generalized random polynomial p such that*

$$\|p(\omega,\cdot) - f(\omega,\cdot)\|_{K(\omega)} < \varepsilon(\omega)$$

for all ω outside a measurable set $L \subset \Omega$ such that $\mu(L) = 0$.

Proof. We denote by $\{K_j\}_{j=0}^{\infty}$ the values taken by K. We also denote

$$\Omega_j := \{\omega \in \Omega : K(\omega) = K_j\} = K^{-1}(K_j).$$

As this is the pre-image of a closed set (the single compact set), each Ω_j is measurable and $\Omega = \bigcup_j \Omega_j$.

We define the measurable subsets of Ω_j to be the σ-algebra $\mathcal{A}_j := \{P \cap \Omega_j : P \in \mathcal{A}\}$. But Ω_j is measurable in Ω, and thus this definition is equivalent to $\mathcal{A}_j = \{P \subset \Omega_j : P \in \mathcal{A}\}$. Most importantly, every measurable set of Ω_j is measurable in Ω.

Let f_j be the restriction of f to $\Omega_j \times K_j$. We claim this is a measurable function. Fix $z \in K_j$. We know that $f(\cdot,z)$ is a measurable mapping between $K^{-1}(z)$ and $\mathbb{C}$ and, for an open set O,

$$(f_j(\cdot,z))^{-1}(O) = \Omega_j \cap (f(\cdot,z))^{-1}(O).$$

This proves that $f(\cdot,z)$ is a measurable function over Ω_j, and thus f is a random function.

By Proposition 1.2, the mapping $\omega \mapsto f(\omega,\cdot)$ between Ω_j and $C(K_j,\mathbb{C})$ is measurable.

We know that $f_j(\omega,\cdot)$ is holomorphic in a neighborhood of K_j. But K_j is, by hypothesis, polynomially convex. Thus, by the Oka–Weil approximation theorem, the multivalued mapping

$$\psi_j(\omega) := \{q \in P(\mathbb{C}^n) : \|f_j(\omega,\cdot) - q(\cdot)\|_{K_j} < \varepsilon(\omega)\}$$

is never empty.

By [1, Theorem 1], and since $P(\mathbb{C}^n)$ is a Suslin subset of $C(K_j,\mathbb{C})$, this means there exists a measurable selection $p_j : \Omega_j \to P(\mathbb{C}^n)$ which approximates f_j to within $\varepsilon(\omega)$ almost everywhere, that is, outside of a measurable set of measure 0.

We now define $p : \Omega \to P(\mathbb{C}^n)$ by setting $p|_{\Omega_j} = p_j$. Clearly, p approximates f almost everywhere. Indeed, if we denote by L_j the set of $\omega \in \Omega_j$ for which p does not approximate f, we deduce that $L = \bigcup_{j=0}^{\infty} L_j$ and thus $\mu(L) = \sum_{i=0}^{\infty} \mu(L_j) = 0$. Since the L_j are measurable, L is measurable.

It remains to show that p is a generalized random function. By Proposition 1.2, we find that for fixed $z \in K_j$, the mapping $\omega \mapsto p_j(\omega,z)$ is measurable. Thus, for a given open set $O \subset \mathbb{C}$ and $z \in \mathbb{C}^n$, the set

$$(p(\cdot,z))^{-1}(O) = \bigcup_{\{j\in\mathbb{N}:\, z\in K_j\}} (p_j(\cdot,z))^{-1}(O)$$

is a measurable set, as a countable union of measurable sets.

Since this is a subset of the measurable set

$$K^{-1}(z) = \bigcup_{j:\, z\in K_j} \Omega_j,$$

the function p is measurable on $K^{-1}(z)$ and hence p is a generalized random function. ∎

It would be interesting to see what happens when trying to remove the "almost everywhere" or by reducing the hypothesis that the set is compact. Could it be true if it were only uniformly separable, or even only measurable?

The following corollary is the Oka–Weil theorem, but since the Oka–Weil theorem was invoked in the proof of Theorem 5.12, the corollary should be considered more appropriately as a special case.

Corollary 5.13. *Let $K \subset \mathbb{C}^n$ be a polynomially convex compact set and f be a measurable function in a neighborhood of K such that $f(\omega,\cdot)$ is holomorphic in a neighborhood of $K(\omega)$ for each ω, and let ε be a positive measurable function defined on Ω. Then there exists a generalized random polynomial p such that*

$$\|p(\omega,\cdot) - f(\omega,\cdot)\|_K < \varepsilon(\omega)$$

for all ω outside a measurable set $L \subset \Omega$ such that $\mu(L) = 0$.

Proof. We apply the theorem to the constant random compact set

$$K(\omega) := K, \quad \forall\omega. \blacksquare$$

COROLLARY 5.14 (Oka–Weil). *Let $K \subset \mathbb{C}^n$ be a polynomially convex compact set and f be a function holomorphic in a neighborhood of K. Then, for each $\varepsilon > 0$, there exists a polynomial p such that*

$$\|p - f\|_K < \varepsilon.$$

Proof. Let $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{B}, \lambda)$ be the probability space $[0,1]$ with Lebesgue measure λ on the Borel algebra $\mathcal{B}$. Now, we apply the previous corollary to the constant measurable function $\varepsilon(\omega) := \varepsilon$. Since $[0,1]$ has positive measure, there is some $\omega_0 \in [0,1]$ such that

$$\|p(\omega_0, \cdot) - f\|_K < \varepsilon.$$

To conclude the proof, we set $p(z) = p(\omega_0, z)$. $\blacksquare$

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References

[1] G. F. Andrus and L. Brown, *Uniform approximation by random functions*, Rocky Mountain J. Math. 14 (1984), 373–381.

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[3] T. Bloom and N. Levenberg, *Random polynomials and pluripotential-theoretic extremal functions*, Potential Anal. 42 (2015), 311–334.

[4] L. Brown and B. M. Schreiber, *Approximation and extension of random functions*, Monatsh. Math. 107 (1989), 111–123.

[5] L. Brown and B. M. Schreiber, *Stochastic continuity and approximation*, Studia Math. 121 (1996), 15–33.

[6] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, NJ, 1969.

[7] C. J. Himmelberg, *Measurable relations*, Fund. Math. 87 (1975), 53–72.

[8] C. J. Himmelberg and F. S. Van Vleck, Some selection theorems for measurable functions, Canad. J. Math. 21 (1969), 394–399.

[9] W. Kaplan, *Introduction to Analytic Functions*, Addison-Wesley, Reading, MA, 1966.

[10] K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 397–403.

[11] P. Malliavin, *Integration and Probability*, Grad. Texts in Math. 157, Springer, New York, 1995.

[12] G. Matheron, *Random Sets and Integral Geometry*, Wiley, New York, 1975.

[13] I. Molchanov, *Theory of Random Sets*, 2nd ed., Probab. Theory Stoch. Model. 87, Springer, London, 2017.

[14] S. Nivoche, *The pluricomplex Green function, capacitative notions, and approximation problems in* $\mathbb{C}^n$, Indiana Univ. Math. J. 44 (1995), 489–510.

[15] A. Nowak, *Characterization of Carathéodory functions*, Ann. Math. Sil. 27 (2013), 93–98.
[16] O. Onicescu and V. I. Istrăţescu, *Approximation theorems for random functions. II*, Rend. Mat. (6) 11 (1978), 585–589.
[17] R. Pemantle and M. C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge Stud. Adv. Math. 140, Cambridge Univ. Press, Cambridge, 2013.
[18] R. M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Grad. Texts in Math. 108, Springer, New York, 1986.
[19] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.
[20] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.
[21] B. M. Schreiber and M. V. Velasco, *Topological algebras of random elements*, Studia Math. 233 (2016), 101–117.
[22] S. M. Srivastava, *A Course on Borel Sets*, Springer, New York, 1998.

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